Level 5 Module 1
exemplar with comments

2020-DSE
MATH EP
M1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2020

Please stick the barcode label here.

Candidate Number									

MATHEMATICS Extended Part
Module 1 (Calculus and Statistics)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9 and 11.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers should be either exact or given to 4 decimal places.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A031E001

SECTION A (50 marks)

1. The table below shows the probability distribution of a discrete random variable X, where a is a constant and $p > 0$.

x	0	1	2
$P(X = x)$	$1-4p$	ap	p

(a) Find a.

(b) If $\text{Var}(2X + a^2) = 8E(aX - 1)$, find p.

(6 marks)

a) $1 - 4p + ap + p = 1$

$p(a-3) = 0$

$a - 3 = 0$

$a = 3$

b) $E(X) = (0)(1-4p) + (1)(3p) + (2)(p)$

$= 5p$

$E(X^2) = (0^2)(1-4p) + (1^2)(3p) + (2^2)(p)$

$= 7p$

$Var(x) = E(X^2) - [E(x)]^2$

$= 7p - (5p)^2$

$= -25p^2 + 7p$

$Var(2X + 3^2) = 8E(3X - 1)$

$2^2 \, Var(X) = 8[3E(X) - 1]$

$4(-25p^2 + 7p) = 120p - 8$

$25p^2 + 23p - 2 = 0$

$(25p - 2)(p + 1) = 0$

$p = 0.08$ or $p = -1$ (rejected)

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2. Using photocopier X, the probability that there is dirt on a page of photocopy is $\frac{1}{5}$. A document of 6 pages is photocopied repeatedly by using X.

(a) Find the probability that there is dirt on a photocopy of the document.

(b) A photocopy of the document is called *acceptable* if there is dirt on fewer than 3 pages, otherwise the photocopy is called *unacceptable*. $\geqslant 3$

(i) Find the probability that a photocopy of the document is *unacceptable*.

(ii) If an *unacceptable* photocopy of the document is just produced, find the expected number of *acceptable* photocopies produced between this *unacceptable* photocopy and the next *unacceptable* photocopy.

(6 marks)

a) Required probability

$= 1 - (1-\frac{1}{5})^6$

$= 0.737856$ //

bi) Required probability

$= (\frac{1}{5})^3(1-\frac{1}{5})^3 C^6_3 + C^6_2(\frac{1}{5})^4(1-\frac{1}{5})^2 + C^6_1(\frac{1}{5})^5(1-\frac{1}{5}) + (\frac{1}{5})^6$

$= 0.09888$ //

bii) Expected number

$= \frac{1}{0.09888} - 1$

$= 9.1133$ //

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$\frac{P(A \cap B)}{P(A)}$

3. Let A and B be two events. Denote the complementary event of A by A'. It is given that $P(B \mid A) = \frac{1}{2}$ and $P(B) = \frac{1}{3} + P(A)$. Suppose that $P(A' \cap B) = kP(A)$, where k is a constant.

(a) By considering $P(A \cap B)$, or otherwise, prove that $k \neq \frac{1}{2}$. Also express $P(B)$ in terms of k.

(b) Are A and B mutually exclusive? Explain your answer.

(c) If A and B are independent, find k.

(7 marks)

a) Denote $P(A)$ by a

$P(B \mid A) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{2}a$

$P(B) = P(A \cap B) + P(A' \cap B) = \frac{1}{3} + P(A)$

$\frac{1}{2}a + ka = \frac{1}{3} + a$

$a(k - \frac{1}{2}) = \frac{1}{3}$

If $k = \frac{1}{2}$, $a = 0$, which is impossible

and k must $> \frac{1}{2}$, so a is positive

$\therefore k \neq \frac{1}{2}$ and $a = \frac{2}{3(2k-1)}$

$P(B) = \frac{1}{2}a + ka$

$= a(\frac{1}{2} + k)$

$= \left[\frac{2}{3(2k-1)}\right](\frac{1}{2} + k)$ //

b) $P(A) \times P(B) = a(\frac{1}{3} + a)$

$= \frac{a}{3} + a^2$

$\neq \frac{1}{2}a$

$= P(A \cap B)$

$\therefore$ A and B are not mutually exclusive //

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c) $P(A) \times P(B) = P(A \cap B)$

$a \times \left[\frac{2}{3(2k-1)}\right]\left(\frac{1}{2}+k\right) = \frac{1}{2}a$

$4\left(\frac{1}{2}+k\right) = 3(2k-1)$

$2+4k = 6k-3$

$2k = 5$

$k = \frac{5}{2}$

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4. A magazine publisher wants to estimate the proportion p of its current subscribers who will continue to subscribe next year. It is given that the publisher randomly selects 841 current subscribers, and 441 of them will continue to subscribe.

(a) Find an approximate 95% confidence interval for p.

(b) An approximate $\beta\%$ confidence interval for p is now constructed. The width of the confidence interval is 0.088. Find β correct to the nearest integer.

(6 marks)

a) an estimate of $p = \frac{441}{841}$

95% confidence interval for p

$$\approx \left(\frac{441}{841} - 1.96 \times \sqrt{\frac{\frac{441}{841}(1-\frac{441}{841})}{841}}, \frac{441}{841} + 1.96 \times \sqrt{\frac{\frac{441}{841}(1-\frac{441}{841})}{841}} \right)$$

$\approx (0.4906, 0.5581)$

b)

$$2z \times \sqrt{\frac{\frac{441}{841}(1-\frac{441}{841})}{841}} < 0.088$$

$$z_1 < 2.555\ 038\ 095$$

$\therefore \beta = (0.4946 \times 2) \times 100$

$= 99$ (cor. to the nearest integer)

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5. Let $f(x)=(1+ke^x)^3$, where k is a real constant.

(a) Express, in terms of k , the constant term and the coefficient of x^2 in the expansion of $f(x)$.

(b) If the constant term in the expansion of $f(x)$ is 27 , find the coefficient of x^2 in this expansion.

(6 marks)

a) $f(x) = (1+ke^x)^3$

$= 1 + C^3_1(1)^2(ke^x) + C^3_2(1)(ke^x)^2 + C^3_3(ke^x)^3$

$= 1 + 3ke^x + 3k^2e^{2x} + k^3e^{3x}$

$= 1 + 3k(1+x+\frac{x^2}{2}+\ldots) + 3k^2(1+2x+\frac{(2x)^2}{2}+\ldots)$

$+ k^3(1+3x+\frac{(3x)^2}{2}+\ldots)$

$= 1+3k+3kx+\frac{3k}{2}x^2+3k^2+6k^2x+6k^2x^2$

$+ k^3+3k^3x+\frac{9k^3}{2}x^2+\ldots$

$= (1+3k+3k^2+k^3)+(3k+6k^2+3k^3)x+(\frac{3k}{2}+6k^2+\frac{9k^3}{2})x^2+\ldots$ //

b) $1+3k+3k^2+k^3 = 27$

$k^3+3k^2+3k-26=0$

$k=2$ or $k=-\frac{5}{2}$ (rejected)

coefficient of x^2

$= \frac{3(2)}{2} + 6(2)^2 + \frac{9(2)^3}{2}$

$= 63$ //

Answers written in the margins will not be marked.

6. Define $g(x) = x + \frac{5}{x} + \ln x^4$ for all non-zero real numbers x.

(a) Find $g'(x)$.

(b) Someone claims that the maximum value of $g(x)$ is less than the minimum value of $g(x)$. Do you agree? Explain your answer.

(c) Write down the equations of the two horizontal tangents to the graph of $y = g(x)$.

(6 marks)

a) $g(x) = x + \frac{5}{x} + \ln x^4$

$= x + \frac{5}{x} + 4\ln x$

$g'(x) = 1 + (5)(-1)(x^{-2}) + 4\left(\frac{1}{x}\right)$

$= 1 + \frac{4}{x} - \frac{5}{x^2}$ //

b) consider $g'(x) = 0$

$1 + \frac{4}{x} - \frac{5}{x^2} = 0$

$x^2 + 4x - 5 = 0$

$x = 1$ or $x = -5$

x	$x < -5$	$x = -5$	$-5 < x < 1$	$x = 1$	$x > 1$
$g'(x)$	+ve	0	−ve	0	+ve

$g(x)$ attains maximum when $x = -5$

maximum value $= (-5) + \frac{5}{(-5)} + \ln(-5)^4$

$= \ln 625 - 6$

$g(x)$ attains minimum when $x = 1$

minimum value $= (1) + \frac{5}{(1)} + \ln(1)^4$

$= 6$

$\therefore$ maximum value < minimum value

$\therefore$ I agree //

c) $y = \ln 625 - 6$

$y = 6$

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7. The total surface area of a solid right circular cylinder is $486\pi\ \text{cm}^2$.

(a) Let $V\ \text{cm}^3$ and r cm be the volume and the base radius of the circular cylinder respectively. Find $\frac{dV}{dr}$.

(b) Can the volume of the circular cylinder exceed $5000\ \text{cm}^3$? Explain your answer.

(6 marks)

a) $\pi r^2 + \pi r \ell = 486\pi$

$r(r+\ell) = 486$

$\ell = \frac{486}{r} - r$

Let h cm be the height of the cylinder

$r^2 + h^2 = \ell^2$

$r^2 + h^2 = \left(\frac{486}{r}\right)^2 - 972 + r^2$

$h = \sqrt{\left(\frac{486}{r}\right)^2 - 972}$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dr} = \left(\frac{1}{3}\pi\right)\left\{(r^2)\left(\frac{1}{2}\right)\left[\left(\frac{486}{r}\right)^2 - 972\right]^{-\frac{1}{2}}\left(\frac{-472392}{r^3}\right) + \left[\left(\frac{486}{r}\right)^2 - 972\right]^{\frac{1}{2}}(2r)\right\}$

$= \frac{1}{3}\pi\left[\left(\frac{486}{r}\right)^2 - 972\right]^{-\frac{1}{2}}\left[\frac{(486)^2}{r} - 1944r\right]$

$= \frac{\pi\left[(486)^2 - 1944r^2\right]}{3r\sqrt{\left(\frac{486}{r}\right)^2 - 972}}$

b) consider $\frac{dV}{dr} = 0$

$r = 11.0227\ 0384$ or $-11.0227\ 0384$ (rejected)

r	$0 < r < 11.0227\ 0384$	$r = 11.02270384$	$r > 11.02270384$
$\frac{dV}{dt}$	+ve	0	−ve

Volume attains maximum when $r = 11.02270384$

maximum volume $= \frac{1}{3}\pi r^2 h$

$= 3966.77921\ \text{cm}^3$

$< 5000\ \text{cm}^3$ $\therefore$ No

Answers written in the margins will not be marked.

8. Let m be a non-zero constant.

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}(xe^{mx})$, find $\int xe^{mx}\,\mathrm{d}x$.

(b) If the area of the region bounded by the curve $y = xe^{mx}$, the x-axis and the straight line $x = 1$ is $\frac{1}{m}$, find m.

(7 marks)

a) $\frac{d}{dx}(xe^{mx}) = (x)(me^{mx}) + (e^{mx})$

$= xme^{mx} + e^{mx}$

$xe^{mx} = \left[\frac{d}{dx}(xe^{mx}) - e^{mx}\right] \times \frac{1}{m}$

$\int xe^{mx}\,dx = \frac{1}{m}\left[xe^{mx} - \int e^{mx}\,dx\right]$

$= \frac{1}{m}\left(xe^{mx} - \frac{e^{mx}}{m}\right) + \text{constant}$

$= \frac{xe^{mx}}{m} - \frac{e^{mx}}{m^2} + \text{constant}$ "

b) consider $y = 0$

$x = 0$

The area

$\int_0^1 xe^{mx}\,dx = \frac{1}{m}$

$\left[\frac{xe^{mx}}{m} - \frac{e^{mx}}{m^2}\right]_0^1 = \frac{1}{m}$

$\frac{e^m}{m} - \frac{e^m}{m^2} + \frac{1}{m^2} = \frac{1}{m}$

$me^m - e^m + 1 = m$

$e^m(m-1) = m - 1$

$e^m = 1$

$m = 0$ "

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SECTION B (50 marks)

9. Every morning Tom leaves home at 7:10 and walks to a certain bus stop to catch bus. He catches the earliest departing bus when he arrives at the bus stop. The time taken for Tom's walk follows a normal distribution with a mean of 15 minutes and a standard deviation of 2 minutes. There are two buses each morning, departing at 7:23 and 7:30 respectively.

(a) Find the probability that Tom catches the bus departing at 7:23 on a certain morning. (2 marks)

(b) Find the probability that Tom catches the bus departing at 7:30 on a certain morning. (1 mark)

(c) Every morning Mary, Tom's student, walks to the same bus stop to catch bus. She catches the earliest departing bus when she arrives at the bus stop. It is given that the probability that Mary catches the bus departing at 7:23 on a certain morning is 0.3015, while the probability that she catches the bus departing at 7:30 on a certain morning is 0.6328. If Mary and Tom catch the same bus, Mary will greet Tom.

(i) Find the probability that the 4th day in a week is the 2nd time Mary greets Tom.

(ii) Given that Mary greets Tom on 2 certain mornings, find the probability that Mary and Tom catch the bus departing at 7:30 on these 2 mornings.

(iii) Given that Mary greets Tom on 4 certain mornings, find the probability that Mary and Tom catch the bus departing at 7:23 on at least 1 of these 4 mornings.

(iv) If Tom wants to have a higher chance of catching the bus departing at 7:23 than that of Mary, what is the latest time for him to leave home? Give your answer correct to the nearest minute. (10 marks)

a) $P\left(Z < \frac{13-15}{2}\right) = P(Z < -1)$

$= 0.1587$

b) $P\left(\frac{13-15}{2} < Z < \frac{20-15}{2}\right) = P(-1 < Z < 2.5)$

$= 0.8351$

ci) Probability that Mary will greet Tom

$= (0.1587)(0.3015) + (0.8351)(0.6328)$

≈ 0.57629933

Required probability

$\approx (0.57629933)(1-0.57629933)^2 C_1^3 (0.57629933)$

≈ 0.178869291

≈ 0.1789

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c(ii) Required probability

$$\doteq \frac{[(0.8351)(0.6328)]^2}{(0.57629933)^2}$$

$$\approx 0.84084061$$

$$\approx 0.8408$$

c(iii) Required probability

$$\cancel{= 1 - \frac{[(0.8351)(0.6328)]^4}{(0.57629933)^4}}$$

$$\cancel{= 0.2930}$$

$$\doteq \frac{[(0.1587)(0.3015)]^4 + [(0.1587)(0.3015)]^3[(0.8351)(0.6328)]C^4_1 + [(0.1587)(0.3015)]^2[(0.8351)(0.6328)]^2 C^4_2 + [(0.1587)(0.3015)]^1[(0.8351)(0.6328)]^3 C^4_1}{(0.57629933)^4}$$

$$= 0.2930$$

c(iv) let n be the minutes before 7:23

$$P\left(Z < \frac{n-15}{2}\right) > 0.3015$$

$$\frac{n-15}{2} > 0.52$$

$$n > 16.04$$

$\therefore$ The latest time for him to leave home is 7:06

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10. A shopping mall launches a campaign to celebrate its fifth anniversary of the opening. A customer can throw a fair die 4 times to receive cash coupons. For each throw, a cash coupon is awarded according to the following table:

Result	1 , 2 or 3	4 or 6	5
Value of cash coupon	\$10	\$25	\$50

(a) Find the probability that a customer receives cash coupons of a total value \$200 . (1 mark)

(b) Find the probability that a customer receives cash coupons of a total value not less than \$150 . (2 marks)

(c) A customer who receives cash coupons of a total value not less than \$150 can join a game. In the game, the customer presses a button 3 times. A number of cakes will show up on a screen for each press of the button, and the number of cakes shown follows a Poisson distribution with a mean of 5 . The result of each press of button is classified as follows:

Number of cakes	1 to 4	5	otherwise
Result	*Good*	*Excellent*	*Fair*

The customer receives a prize according to the following results:

Result	1 *Excellent* and 2 *Good*	2 *Excellent* and 1 *Good*	3 *Excellent*
Prize	a cup	a backpack	an oven

(i) Find the probability that a customer joining the game receives a backpack.

(ii) Given that a customer joining the game receives a prize, find the probability that the customer receives an oven.

(iii) A customer who cannot join the game can still receive a cup by joining a lucky draw. In the lucky draw, the probability of receiving a cup is 0.01 . Given that a customer receives a cup, find the probability that the customer cannot join the game. (9 marks)

a) Required probability

$= \left(\frac{1}{6}\right)^4$

$= \frac{1}{1296}$ //

b) Required probability

$= \frac{1}{1296} + \left(\frac{1}{6}\right)^3\left(\frac{2}{6}\right) C^4_3 + \left(\frac{1}{6}\right)^3\left(\frac{3}{6}\right) C^4_3 + \left(\frac{1}{6}\right)^2\left(\frac{2}{6}\right)^2 C^4_2$

$= \frac{37}{1296}$ //

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c i) $P(\text{Good}) = e^{-5}\left(5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right)$

$\approx 0.433\,755\,338$

$P(\text{Excellent}) = e^{-5}\left(\frac{5^5}{5!}\right)$

$\approx 0.175\,467\,369$

$P(\text{Fair}) \approx 1 - P(\text{Good}) - P(\text{Excellent})$

$\approx 1 - 0.433\,755\,338 - 0.175\,467\,369$

$\approx 0.390\,777\,292$

Required probability

$\approx (0.175\,467\,369)^2(0.433\,755\,338)\,C_1^3$

$\approx 0.040\,064\,415$

c ii) Required probability

$$= \frac{(0.175\,467\,369)^3}{(0.175\,467\,369)(0.433\,755\,338)^2 C_1^3 + 0.040\,064\,415 + (0.175\,467\,369)^3}$$

$\approx 0.037\,385\,48$

≈ 0.0374

c iii) Required probability

$$= \frac{\left(1 - \frac{37}{1296}\right)(0.01)}{\left(\frac{37}{1296}\right)(0.175\,467\,369)(0.433\,755\,338)^2 C_1^3 + \left(1 - \frac{37}{1296}\right)(0.01)}$$

≈ 0.7746

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11. The manager of a park models the rate of change of the number of adults (in thousand per month) visiting the park by

$$A(t) = 5\ln(t^2 - 8t + 20) \ ,$$

where t is the number of months elapsed since the park opens. Denote the total number of adults visiting the park in the first 2 months since the park opens by α thousand. Let α_1 be the estimate of α by using the trapezoidal rule with 5 sub-intervals.

(a) (i) Find α_1 .

(ii) Is α_1 an over-estimate or an under-estimate? Explain your answer.

(5 marks)

(b) The manager models the rate of change of the number of children (in thousand per month) visiting the park by

$$B(t) = \frac{3^{2t+2}}{1+3^{2t}} \ ,$$

where t is the number of months elapsed since the park opens.

(i) Find the total number of children visiting the park in the first 2 months since the park opens.

(ii) The manager claims that in the first 2 months since the park opens, the difference of the total number of adults visiting the park and the total number of children visiting the park exceeds 40% of the total number of adults visiting the park. Do you agree? Explain your answer.

(7 marks)

a i) $\alpha_1 = \left(\frac{1}{2}\right)\left(\frac{2-0}{5}\right)\left\{A(0)+A(2)+2\left[A(0.4)+A(0.8)+A(1.2)+A(1.6)\right]\right\}$

$\approx 25.548\ 550\ 95$ thousand

a ii) $A(t) = 5\ln(t^2-8t+20)$

$A'(t) = (5)\left(\frac{1}{t^2-8t+20}\right)(2t-8)$

$= \frac{10(t-4)}{t^2-8t+20}$

$A''(t) = \frac{10\left[(t^2-8t+20)-(t-4)(2t-8)\right]}{(t^2-8t+20)^2}$

$= \frac{-10t^2+80t-120}{(t^2-8t+20)^2}$

$A''(t) < 0$ for $0 \le t \le 2$

$\therefore$ $A(t)$ opens downwards

$\therefore$ α_1 is an under-estimate

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bi) Total number of children

$= \int_0^2 B(t)\, dt$

$= \int_0^2 \frac{3^{2t} \cdot 9}{1+3^{2t}}\, dt$ Let $u = 1 + 3^{2t}$

$du = 2\ln 3 (3^{2t})\, dx$

$= \int_2^{82} \frac{\cancel{3^{2t}} \cdot 9}{u} \left(\frac{du}{2\ln 3 (\cancel{3^{2t}})} \right)$ when $t = 2$, $u = 82$

when $t = 0$, $u = 2$

$= \frac{9}{2\ln 3} \int_2^{82} u^{-1}\, du$

$= \frac{9}{2\ln 3} \left[\ln|u| \right]_2^{82}$

$\approx 15.211\ 075\ 35$ thousand //

bii) Difference of the total number of adults and children

$= \int_0^2 A(t)\, dt - \int_0^2 B(t)\, dt$

$> 25.548\ 550\ 95$ thousand $- 15.211\ 075\ 35$ thousand

$= 10.337\ 4756$ thousand

40% of the total number of adults

$= 40\% \times \int_0^2 A(t)\, dt$

$> 40\% \times 25.548\ 550\ 95$ thousand

$\doteq 10.21942038$

$<$ Differerence of the total number of adults and children

$\therefore$ The claim is agreed //

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12. John studies the number of ducks P (in thousand) in a farm by

$$P = \frac{32}{a^{5+bt}+8} ,$$

where a and b are constants and t $(t \geq 0)$ is the number of months elapsed since the start of the study.

(a) Express $\ln\left(\frac{32}{P}-8\right)$ as a linear function of t. (2 marks)

(b) John finds that the graph of the linear function obtained in (a) passes through the point $(1, \ln 2)$ and the intercept on the vertical axis of the graph is $\ln 32$.

(i) Find a and b .

(ii) Find $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

(iii) Estimate the number of ducks in the farm after a very long time. Hence, or otherwise, prove that the number of ducks in the farm does not exceed 4 thousand since the start of the study.

(iv) Find the number of ducks in the farm when $\frac{dP}{dt}$ attains its greatest value.

(11 marks)

a) $P = \frac{32}{a^{5+bt}+8}$

$\frac{32}{P} - 8 = a^{5+bt}$

$\ln\left(\frac{32}{P}-8\right) = bt\ln a + 5\ln a$,,

bi) $5\ln a = \ln 32$

$a^5 = 32$

$a = 2$,,

$b\ln a = \frac{\ln 2 - \ln 32}{1 - 0}$

$b = -4$,,

bii) $P = \frac{32}{2^{5-4t}+8}$

$\frac{dP}{dt} = (32)(-1)(2^{5-4t}+8)^{-2}(2^5)(-4\ln 2)(2^{-4t})$

$= \frac{4096\ln 2\,(2^{-4t})}{(2^{5-4t}+8)^2}$

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Answers written in the margins will not be marked.

b ii)

$$\frac{d^2P}{dt^2} = \frac{(4096\ln 2)\left[(2^{5-4t}+8)^2(-4\ln 2)(2^{-4t}) - (2^{-4t})(2)(2^{5-4t}+8)(2^5)(-4\ln 2)(2^{-4t})\right]}{(2^{5-4t}+8)^4}$$

$$= \frac{(4096\ln 2)(-4\ln 2)(2^{-4t})\left[(2^{5-4t}+8)-(2^{6-4t})\right]}{(2^{5-4t}+8)^3}$$

$$= \frac{-16384(\ln 2)^2(2^{-4t})\left[8-32(2^{-4t})\right]}{(2^{5-4t}+8)^3}$$

$$= \frac{-131072(\ln 2)^2(2^{-4t})\left[1-4(2^{-4t})\right]}{(2^{5-4t}+8)^3}$$

b iii) Number of ducks after a very long time

$= \lim_{t\to\infty} P$

$= \lim_{t\to\infty} \frac{32}{(0)+8}$

$= 4$ thousand

$\frac{dP}{dt} > 0$ for $t \geq 0$

$\therefore$ P is a continuous increasing function,

P's maximum is 4 thousand

$\therefore$ It won't exceed 4 thousand

b iv) consider $\frac{d^2P}{dt^2} = 0$

$1 - 4(2^{-4t}) = 0$

$2^{-4t} = \frac{1}{4}$

$-4t\ln 2 = \ln\frac{1}{4}$

$t = \frac{1}{2}$

$t = \frac{1}{2}$	$0 < t < \frac{1}{2}$	$t = \frac{1}{2}$	$t > \frac{1}{2}$
$\frac{d^2P}{dt^2}$	+ve	0	-ve

$\frac{dP}{dt}$ attains its greatest value when $t = \frac{1}{2}$

→ NPP

Answers written in the margins will not be marked.

The number of ducks when $t = \frac{1}{2}$

$$= \frac{32}{2^{5-4(\frac{1}{2})} + 8}$$

$= 2$ thousand

Answers written in the margins will not be marked.

END OF PAPER

Answers written in the margins will not be marked.

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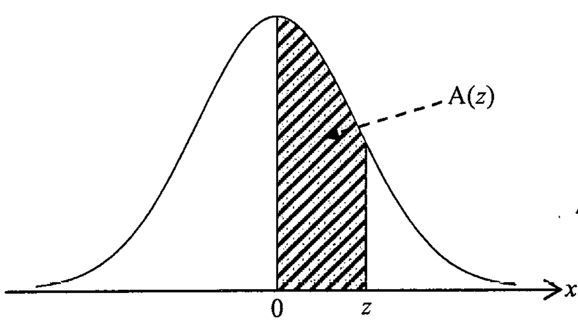
Standard Normal Distribution Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

Note : An entry in the table is the area under the standard normal curve between $x = 0$ and $x = z$ $(z \geq 0)$. Areas for negative values of z can be obtained by symmetry.



$$A(z) = \int_0^z \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

Comments

The candidate demonstrates comprehensive knowledge and understanding of the concepts underpinning calculus and statistics in the curriculum by applying them successfully at a sophisticated level to a wide range of unfamiliar situations, such as in Questions 9, 10 and 12.

He/She is able to communicate and express views and arguments precisely and logically using mathematical language and notations. Typical examples are his/her solutions in Questions 1, 2, 8, 9, 10 and 12.

He/She is also able to formulate mathematical models successfully in complex situations, employ appropriate strategies to arrive at a complete solution, and evaluate the significance and reasonableness of the results obtained, such as in Questions 9 and 12.

It can be concluded that the candidate has the ability to integrate knowledge and skills from different areas of the curriculum in handling complex tasks using a variety of strategies.