2019-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2019

Please stick the barcode label here.

Candidate Number									

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

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FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

1. Let $f(x) = \dfrac{10x}{7+3x^2}$. Prove that $f(1+h) - f(1) = \dfrac{4h-3h^2}{10+6h+3h^2}$. Hence, find $f'(1)$ from first principles. (4 marks)

$$f(1+h) - f(1) = \frac{10(1+h)}{7+3(1+h)^2} - \frac{10(1)}{7+3(1)^2}$$

$$= \frac{10h+10}{3h^2+6h+10} - 1$$

$$= \frac{10h+10-(10+6h+3h^2)}{10+6h+3h^2}$$

$$= \frac{4h-3h^2}{10+6h+3h^2}$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\left[\frac{4h-3h^2}{10+6h+3h^2}\left(\frac{1}{h}\right)\right]$$

$$= \lim_{h\to 0}\frac{4-3h}{10+6h+3h^2}$$

$$= \frac{2}{5}$$

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2. Let $P(x) = \begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbf{R}$. It is given that the coefficient of x^3 in the expansion of $P(x)$ is 160. Find

(a) λ,

(b) $P'(0)$.

(5 marks)

(a) $P(x) = (x+\lambda)\left[(x+\lambda)^2(x+\lambda)^3 - 3(5)\right] - \left[0-3(4)\right] + 2\left[0-4(x+\lambda)^2\right]$

$= (x+\lambda)^6 - 15(x+\lambda) + 12 - 8(x+\lambda)^2$

$= x^6 + 6\lambda x^5 + 15\lambda^2x^4 + 20\lambda^3x^3 + 15\lambda^4x^2 + 6\lambda^5x + \lambda^6 - 15x - 15\lambda$
$+ 12 - 8x^2 - 16\lambda x - 8\lambda^2$

$= x^6 + 6\lambda x^5 + 15\lambda^2x^4 + 20\lambda^3x^3 + (15\lambda^4 - 8)x^2 + (6\lambda^5 - 16\lambda - 15)x + \lambda^6 -$
$8\lambda^2 - 15\lambda + 12$

$\therefore \quad 20\lambda^3 = 160$

$\lambda = 2.$

(b). $P'(0)$

$= 6(0)^5 + 30\lambda(0)^4 + 60\lambda^2(0)^3 + 60\lambda^3(0)^2 + 2(15\lambda^4 - 8)(0) + (6\lambda^5 - 16\lambda - 15)(1)$

$= 6\lambda^5 - 16\lambda - 15$

$= 6(2)^5 - 16(2) - 15$

$= 145$

Answers written in the margins will not be marked.

3. A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580 cm^3 of liquid X. The researcher finds that during the experiment, $\frac{dV}{dt} = -2t$, where $V\text{ cm}^3$ is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

(a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

(b) It is given that $V = h^2 + 24h$, where $h\text{ cm}$ is the depth of liquid X in the vessel. Find the value of $\frac{dh}{dt}$ when $t = 18$.

(6 marks)

(a) $V = \int -2t\, dt$

$= -t^2 + c$

Put $t=0$, $V = 580$

$580 = -0^2 + c$

$c = 580$.

$\therefore V = 580 - t^2$.

Put $t = 24$

$V = 580 - (24)^2$

$= 4$

> 0.

$\therefore$ The claim is correct.

(b) $V = 580 - (18)^2$

$= 256\text{ cm}^3$

$V = h^2 + 24h$

$256 = h^2 + 24h$

$h = 8$ or -32 (rej).

$V = h^2 + 24h$

$\frac{dV}{dt} = (2h+24)\left(\frac{dh}{dt}\right)$

$-2(18) = [2(8)+24]\frac{dh}{dt}$

$\frac{dh}{dt} = -0.9$

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4. Define $g(x)=\dfrac{\ln x}{\sqrt{x}}$ for all $x\in(0,99)$. Denote the graph of $y=g(x)$ by G .

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G , the x-axis and the vertical line passing through the maximum point of G . Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

(a) $g'(x) = \dfrac{\sqrt{x}\left(\frac{1}{x}\right) - \ln x\left(\frac{1}{2}\right)\left(x^{-\frac{1}{2}}\right)}{(\sqrt{x})^2}$

$= \dfrac{2-\ln x}{2x^{\frac{3}{2}}}$

$g'(x) = 0.$

$2-\ln x = 0.$

$x = e^2.$

x	$0<x<e^2$	$x=e^2$	$e^2<x<99$
$g'(x)$	$+ve$	0	$-ve.$

$\therefore$ The only maximum point of G is $\left(e^2, \frac{2}{e}\right)$

(b) Put $y=0$ into $y=g(x)$

$0 = \dfrac{\ln x}{\sqrt{x}}$

$x=1.$

Volume of solid $= \pi\int_1^{e^2}\left(\dfrac{\ln x}{\sqrt{x}}\right)^2 dx$

$= \pi\int_1^{e^2}(\ln x)^2\, d(\ln x)$

$= \pi\left[\dfrac{(\ln x)^3}{3}\right]_1^{e^2}$

$= \pi\left(\dfrac{8}{3}-0\right)$

$= \dfrac{8\pi}{3}.$

Answers written in the margins will not be marked.

5. (a) Using mathematical induction, prove that $\displaystyle\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$ for all positive integers n.

(b) Using (a), evaluate $\displaystyle\sum_{k=50}^{200} \frac{1}{k(k+1)}$.

(7 marks)

5a. When $n=1$

$$\text{L.H.S} = \sum_{k=1}^{2} \frac{1}{k(k+1)}$$
$$= \frac{1}{1(1+1)} + \frac{1}{2(2+1)}$$
$$= \frac{2}{3}.$$
$$\text{R.HS} = \frac{1+1}{1(2+1)}$$
$$= \frac{2}{3}.$$

LHS = R.HS

$\therefore$ The statement is true for $n=1$

Assume that the statement is also true for $n=p$, where p is a positive integer.

i.e. $\displaystyle\sum_{k=p}^{2p} \frac{1}{k(k+1)} = \frac{p+1}{p(2p+1)}$

when $n=p+1$

$$\text{L.H.S} = \sum_{k=p+1}^{2(p+1)} \frac{1}{k(k+1)}$$

$$= \sum_{k=p}^{2p} \frac{1}{k(k+1)} - \frac{1}{p(p+1)} + \frac{1}{(2p+1)[2(2p+1)+1]} + \frac{1}{(2p+2)[2(2p+2)+1]}$$
$$= \frac{p+1}{p(2p+1)} - \frac{2}{p(p+1)} + \frac{1}{(2p+1)(4p+3)} + \frac{1}{2(p+1)(4p+5)}$$
$$= \frac{(p+1)^2(2)(4p+3)(4p+5) - 4(2p+1)(4p+3)(4p+5) + 2p(2p+1)(4p+5) + p(2p+1)(4p+3)}{2p(p+1)(2p+1)(4p+3)(4p+5)}$$
$$= \frac{32p^4 + 24p^3 - 92p^2 - 111p - 30}{2p(p+1)(2p+1)(4p+3)(4p+5)}$$
$$= \frac{2p+1+1}{(2p+1)(2p+1+1)}$$
$$= \text{R.HS}$$

The statement is true for $n=p+1$

By the principle of mathematical induction, the statement is true for all positive integers n.

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(b) $\displaystyle\sum_{k=50}^{200} \frac{1}{k(k+1))}$

$= \displaystyle\sum_{k=50}^{100} \frac{1}{k(k+1)} + \sum_{k=100}^{200} \frac{1}{k(k+1)} - \frac{1}{100(100+1)}$

$= \dfrac{50+1}{50(2(50)+1)} + \dfrac{100+1}{100(2(100)+1)} - \dfrac{1}{100(101)}$

$= \dfrac{151}{10050}$.

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6. Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x - 2y - 2z = \beta \\ 5x + \alpha y + \alpha z = 5\beta \\ 7x + (\alpha-3)y + (2\alpha+1)z = 8\beta \end{cases},\ \text{where } \alpha, \beta \in \mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α.

(ii) Express y in terms of α and β.

(b) Assume that $\alpha = -4$. If (E) is inconsistent, find the range of values of β.

(7 marks)

(a)(i) $\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix} \neq 0$.

$a(2a+1) - \alpha(a-3) + 2[5(2\alpha+1) - 7a] - 2[5(\alpha-3) - 7a] \neq 0$.

$2a^2 + a - a^2 + 3a + 6a + 10 + 4a + 30 \neq 0$.

$a^2 + 14a + 40 \neq 0$.

$(\alpha-4)(\alpha-10) \neq 0$

$\alpha \neq 4$ and $\alpha \neq 10$.

The range of α is all real numbers except $\alpha = 4$ and $\alpha = 10$.

(a)(ii) $y = \dfrac{\Delta y}{\Delta}$

$= \dfrac{\begin{vmatrix} 1 & \beta & -2 \\ 5 & 5\beta & \alpha \\ 7 & 8\beta & 2\alpha+1 \end{vmatrix}}{\alpha^2 + 14\alpha + 40}$

$= \dfrac{5\beta(2\alpha+1) - 8\alpha\beta - \beta[5(2\alpha+1) - 7\alpha] - 2[5(8\beta) - 5\beta(7)]}{\alpha^2 + 14\alpha + 40}$

$= \dfrac{-\alpha\beta - 10\beta}{\alpha^2 + 14\alpha + 40}$

(b). $\left[\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & -4 & -4 & 5\beta \\ 7 & -7 & -7 & 8\beta \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 6 & 6 & 0 \\ 0 & 7 & 7 & \beta \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{array}\right]$

Answers written in the margins will not be marked.

For (E) to be inconsistent, $\beta \neq 0$.

$\therefore$ The range of values of β is all real numbers except $\beta = 0$.

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7. (a) Using integration by parts, find $\int e^x \sin \pi x \, dx$.

(b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x \, dx$.

(7 marks)

(a) $\int e^x \sin \pi x \, dx$

$= \int \sin \pi x \, d(e^x)$

$= e^x \sin \pi x - \int e^x \, d(\sin \pi x)$

$= e^x \sin \pi x - \pi \int e^x \cos \pi x \, dx$

$= e^x \sin \pi x - \pi \int \cos \pi x \, d(e^x)$

$= e^x \sin \pi x - \pi \left[e^x \cos \pi x - \int e^x \, d(\cos \pi x) \right]$

$= e^x \sin \pi x - \pi e^x \cos \pi x - \pi^2 \int e^x \sin \pi x \, dx$.

$\therefore \int e^x \sin \pi x \, dx = \dfrac{e^x \sin \pi x - \pi e^x \cos \pi x}{\pi^2 + 1} + C$.

(b). Let $u = 3 - x$.

$du = -dx$.

when $x = 3$, $u = 0$.

when $x = 0$, $u = 3$.

$-\int_3^0 e^u \sin \pi (3-u) \, du$

$= \int_0^3 e^u \sin (\pi - \pi u) \, du$

$= \int_0^3 e^u \sin \pi u \, du$

$= \left[\dfrac{e^u \sin \pi u - \pi e^u \cos \pi u}{\pi^2 + 1} \right]_0^3$.

$= \dfrac{e^3}{\pi^2+1} - \left(-\dfrac{\pi}{\pi^2+1} \right)$

$= \dfrac{e^3 + \pi}{\pi^2 + 1}$

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8. Let $h(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. It is given that $h'(x)=\dfrac{2x^2-7x+8}{x}$ for all $x>0$.

(a) Is $h(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y=h(x)$ by H. It is given that H passes through the point $(1,3)$. Find

(i) the equation of H,

(ii) the point(s) of inflexion of H.

(8 marks)

(a) $h'(x)=\dfrac{2x^2-7x+8}{x}$

$=\dfrac{2\left[(x-\frac{7}{4})^2-\frac{49}{16}\right]+8}{x}$

$=\dfrac{2(x-\frac{7}{4})^2+\frac{15}{8}}{x}$

$\geq \dfrac{15}{8x}$

≥ 0

$\therefore$ $h'(x)>0$ for all values of x (where $x>0$).

$\therefore$ $h(x)$ is an increasing function.

(b)(i) $\int \dfrac{2x^2-7x+8}{x}\,dx$

$=\int 2x-7+8x^{-1}\,dx$

$=x^2-7x+8\ln|x|+C$

$\therefore$ $h(x)=x^2-7x+8\ln|x|+C$

$h(1)=3$.

$1^2-7(1)+8\ln 1+C=3$

$C=9$

$\therefore$ The equation of H is $y=x^2-7x+8\ln|x|+9$.

(b)(ii) $h''(x)=\dfrac{x(4x-7)-(2x^2-7x+8)}{x^2}$

$=\dfrac{2x^2-8}{x^2}$

$h''(x)=0$.

$2x^2-8=0$

$x=2$ or -2 (rej)

Answers written in the margins will not be marked.

x	$0 < x < 2$	$x = 2$	$x > 2$.
$h''(x)$	$-$	0.	$+$

$\therefore$ The point of inflexion is $(2, 8\ln 2 - 1)$

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SECTION B (50 marks)

9. Consider the curve Γ: $y = \frac{1}{3}\sqrt{12 - x^2}$, where $0 < x < 2\sqrt{3}$. Denote the tangent to Γ at $x = 3$ by L .

(a) Find the equation of L . (3 marks)

(b) Let C be the curve $y = \sqrt{4 - x^2}$, where $0 < x < 2$. It is given that L is a tangent to C. Find

(i) the point(s) of contact of L and C ;

(ii) the point(s) of intersection of C and Γ ;

(iii) the area of the region bounded by L , C and Γ .

(9 marks)

(a). $\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{2}\right)(12-x^2)^{-\frac{1}{2}}(-2x)$

$= \frac{-x}{3\sqrt{12-x^2}}$

$\left.\frac{dy}{dx}\right|_{x=3} = \frac{-3}{3\sqrt{12-3^2}}$

$= -\frac{1}{\sqrt{3}}$.

Put $x=3$ into $y = \frac{1}{3}\sqrt{12-x^2}$

$y = \frac{1}{3}\sqrt{12-3^2}$

$= \frac{\sqrt{3}}{3}$.

The required eq^{n} is

$\frac{y - \frac{\sqrt{3}}{3}}{x-3} = -\frac{1}{\sqrt{3}}$

$\sqrt{3}y - 1 = -x+3$.

$x + \sqrt{3}y - 4 = 0$.

(b)(i) $\frac{d}{dx}(\sqrt{4-x^2}) = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$

$= \frac{-x}{\sqrt{4-x^2}}$

$\frac{-x}{\sqrt{4-x^2}} = -\frac{1}{\sqrt{3}}$

$\sqrt{3}x = \sqrt{4-x^2}$

$3x^2 = 4 - x^2$

$x = 1$ or -1 (rej)

Points of contact of L and C is $(1, \sqrt{3})$

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(b)(ii) $\frac{1}{3}\sqrt{12-x^2} = \sqrt{4-x^2}$

$12-x^2 = 9(4-x^2)$

$12-x^2 = 36-9x^2$

$8x^2 = 24$

$x^2 = 3$

$x = \sqrt{3}$ or $-\sqrt{3}$ (rej)

The points of intersection of C and Γ is $(\sqrt{3}, 1)$,

(b)(iii) $\int_1^3 \frac{4-x}{\sqrt{3}}\,dx - \int$

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10. (a) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{2+\cos 2x} = \frac{\sec^2 x}{2+\sec^2 x}$. (1 mark)

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x} \, dx$. (3 marks)

(c) Let $f(x)$ be a continuous function defined on $\mathbf{R}$ such that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

Prove that $\int_{-a}^{a} f(x)\ln(1+e^x)\, dx = \int_0^a x f(x)\, dx$ for any $a \in \mathbf{R}$. (4 marks)

(d) Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2} \ln(1+e^x)\, dx$. (5 marks)

(a). $\frac{1}{2+\cos 2x}$

$= \frac{1}{2+2\cos^2 x-1}$

$= \frac{1}{2\cos^2 x+1}$

$= \frac{1}{\cos^2 x\left(2+\frac{1}{\cos^2 x}\right)}$

$= \frac{\sec^2 x}{2+\sec^2 x}$.

(b). $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x} dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{2+\sec^2 x} dx$

$=$

$=$

(c.) $\int_{-a}^{a} f(x)\ln(1+e^x)dx$

$= \int_0^a 2f(x)\ln(1+e^x)dx$.

(d). Let $f(x) = \frac{\sin 2x}{(2+\cos 2x)^2}$

$f(-x) = \frac{\sin 2(-x)}{[2+\cos 2(-x)]^2}$

$= \frac{-\sin 2x}{(2+\cos 2x)^2}$

$= -f(x)$

$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)dx = \int_0^{\frac{\pi}{4}} \frac{x\sin 2x}{(2+\cos 2x)^2} dx$

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(c)

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11. Let $M=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2=aM+bI$. (3 marks)

(b) Prove that $6M^n=(1-(-5)^n)M+(5+(-5)^n)I$ for all positive integers n. (4 marks)

(c) Does there exist a pair of 2×2 real matrices A and B such that $(M^n)^{-1}=A+\frac{1}{(-5)^n}B$ for all positive integers n? Explain your answer. (5 marks)

(a) $M^2=\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}$

$=\begin{pmatrix}-3 & -28\\ 4 & 29\end{pmatrix}$

$M^2=aM+bI$

$\begin{pmatrix}-3 & -28\\ 4 & 29\end{pmatrix}=\begin{pmatrix}2a+b & 7a\\ -a & -6a+b\end{pmatrix}$

$-a=4$

$a=-4$

$2a+b=-3$

$2(-4)+b=-3$

$b=5$

(b). when $n=1$

L.H.S $=6M$

R.H.S. $=(1-(-5))M+(5+(-5))I$

$=6M$

L.H.S = R.H.S

$\therefore$ The statement is true for $n=1$

Assume that the statement is true for $n=k$ where k is a positive integer.

i.e. $6M^k=(1-(-5)^k)M+(5+(-5)^k)I$

when $n=k+1$

L.H.S $=6M^{k+1}$

$=6M^k(M)$

Answers written in the margins will not be marked.

$= [(1-(-5)^k)M + (5+(-5)^k)I]M$

$= (1-(-5)^k)M^2 + (5+(-5)^k)M$

$= (1-(-5)^k)(-4M+5I) + (5+(-5)^k)M$

$= -4M + 5I + 4(-5)^k M - 5(-5)^k I + 5M + (-5)^k M$

$= [1+5(-5)^k]M + [5-5(-5)^k]I$

$= [1-(-5)^{k+1}]M + [5+(-5)^{k+1}]I$

$=$ R.H.S.

The statement is true for $n=k+1$

By the principles of mathematical induction, the statement is true for all positive integers n.

(c).

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12. Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$, where O is the origin and t is a constant. It is given that $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC}\right|$.

(a) Find t . (3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (2 marks)

(c) Find the volume of the pyramid $OABC$. (2 marks)

(d) Denote the plane which contains A , B and C by Π . It is given that P , Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OR} = r\mathbf{k}$. Let D be the projection of O on Π .

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(iii) Let E be a point such that $\overrightarrow{OE} = \frac{1}{p}\mathbf{i} + \frac{1}{q}\mathbf{j} + \frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D , E and O . Explain your answer.

(6 marks)

(a) $|\vec{AC}| = |\vec{BC}|$

$|\vec{OC} - \vec{OA}| = |\vec{OC} - \vec{OB}|$

$\sqrt{(-6)^2 + (-8)^2 + (t-2)^2} = \sqrt{(-8)^2 + (t-8)^2}$

$t^2 - 4t + 104 = t^2 - 16t + 128.$

$12t = 24$

$t = 2.$

(b). $\vec{AB} \times \vec{AC}$

$= (-6\vec{i} + 6\vec{k}) \times (-6\vec{i} - 8\vec{j})$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 0 & 6 \\ -6 & -8 & 0 \end{vmatrix}$

$= 48\vec{i} - 36\vec{j} + 48\vec{k}$

(c). Volume $= \frac{1}{6} \left\| \vec{AO} \cdot \vec{AB} \times \vec{AC} \right\|$

$= \frac{1}{6} \left\| \begin{matrix} -1 & 4 & -2 \\ -6 & 0 & 6. \\ -6 & -8 & 0 \end{matrix} \right\|$

$= 48$

(d)(i) Since O does not lie on plane π, while P, Q, R do lie on plane π, p, q and $r \neq 0$

$\therefore pqr \neq 0.$

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d(ii) $\overrightarrow{OD} = \dfrac{|\overrightarrow{OA} \cdot \overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AB} \times \overrightarrow{AC}|^2} (\overrightarrow{AB} \times \overrightarrow{AC})$

$= \dfrac{288}{5904} (48\vec{i} - 36\vec{j} + 48\vec{k})$

$= \dfrac{96}{41}\vec{i} - \dfrac{72}{41}\vec{j} + \dfrac{96}{41}\vec{k}$

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